Int. J. Solids Structures Vol. 25, No. 5, pp. 553–561, 1989
Printed in Great Britain.

0020-7683/89 $3.00+.00

INTERACTION OF SHEAR WAVES WITH A PERIODIC ARRAY OF RIGID STRIPS

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(*Received* 13 *April* 1988; *in revised form* 26 *October* 1988)

Abstract—The interaction of shear waves with a periodic array of rigid strips is investigated. Using conventional analytical techniques the response of the strips is determined as a function of frequency. High amplifications in the strip response are found to occur at particular frequencies. The effects of spacing-to-width ratios and strip-to-matrix mass ratios on these amplifications are examined.

INTRODUCTION

Elastic wave interaction with an embedded rigid strip has been the topic of several studies during the past two decades. The focus of these studies has been on determining the response of the strip and/or the stress intensification at the ends of the strip as a function of frequency. The interaction of P- and SV-waves with a semi-infinite, rigid strip was investigated by Thau and Pao (1967). Using an exact analytical method, they computed the normal stresses along the boundary of the strip as a function of normalized wavenumber. Their analysis, however, is based on the assumption of smooth contact between the strip and the surrounding medium. Scattering of P- and SH-waves by a perfectly bonded and finite rigid strip was investigated subsequently by Jain and Kanwal (1972). Using an asymptotic approximation, they obtained results for the scattered field valid for low-to-moderately low frequencies. More recently, Meade and Keer (1982) considered a problem similar to that of Jain and Kanwal, although their study is restricted to SH-waves. Using an exact approach, they obtained results valid out to moderately high frequencies. A number of similar studies on the related problem of a rigid strip bonded to an elastic half space have been reported (see e.g. Oien, 1971, where further references are given).

Here we consider an infinite, vertical row of equally spaced rigid strips in perfect contact with a surrounding elastic medium. The array of strips is subjected to plane harmonic SV-waves with propagation vector parallel to the strip orientation. Generalizing the approach of Meade and Keer, we obtain the translation and rotation of the strip array as a function of frequency.

FORMULATION

Figure 1 depicts the geometry of the array of strips. A state of plane strain is assumed in which $(u, v)\,e^{-i\omega t}$ are the displacements and $(\sigma_x, \sigma_y, \tau)\,e^{-i\omega t}$ are the corresponding stress components, where ω is the frequency. Henceforth the time factor $e^{-i\omega t}$ will be omitted. As a consequence of the plane strain assumption, each strip has an infinite length normal to the (x, y) plane. It is assumed that each strip is in perfect and complete contact with the surrounding elastic medium. Moreover, we assume that the strips are sufficiently thin, for mathematical purposes, to be treated as linear inclusions. The elastic constants and mass density of the medium are denoted as (λ, μ) and ρ, respectively, and the mass per unit length of the strips is denoted as ρ^*.

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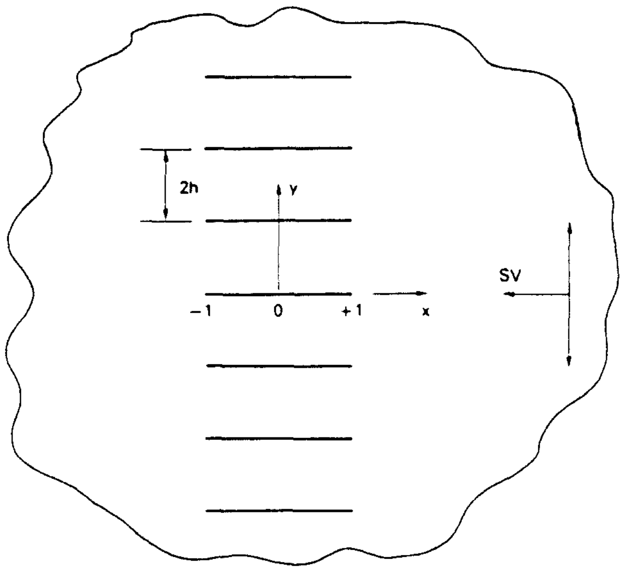

Fig. 1. Geometry of strip array and incident shear wave.

Let a plane harmonic shear wave of the form

$$u_i = 0, \quad v_i = e^{ik_T x}, \tag{1}$$

where k_T is the wave number, be incident on the array of strips. On account of symmetry we may restrict attention to the region $-h < y < h$, $-\infty < x < \infty$. In the sequel the strip situated at $y = 0$ will be referred to as the basic strip. It is easily shown that the disturbance (1) produces a jump in normal stress $[\sigma_y]$ across the basic strip. The shear stress τ, however, as well as the displacements are continuous across the strip. Further, it is clear by symmetry that

$$u = \sigma_y = 0, \quad y = \pm h \tag{2}$$

$$u = 0, \quad y = 0. \tag{3}$$

Using these conditions in conjunction with the basic field equations, it can be shown that the relationship between the vertical displacement v and the jump in normal stress $[\sigma_y]$ across the basic strip is defined by the (singular) integral equation

$$4\pi k_T^2 v(x) = \int_{-1}^{1} [p(s)] K(s,x) \, ds, \quad |x| < 1 \tag{4}$$

where

$$p(s) = \sigma_y(s)/\mu$$

$$K(s,x) = K_L(s,x) + K_T(s,x) \tag{5}$$

$$K_L(s,x) = -i\pi k_L H_1^{(1)}(k_L|w|)/|w| - 2\pi/h \sum_1^\infty \alpha_n^2/\zeta_{nL} \, e^{-\zeta_{nL}|w|} \tag{6}$$

$$K_T(s,x) = -i\pi k_T^2 H_0^{(1)}(k_T|w|) + i\pi k_T H_1^{(1)}(k_T|w|)/|w| + 2\pi/h \sum_1^\infty \zeta_{nT} \, e^{-\zeta_{nT}|w|} \tag{7}$$

in which $w = s-x$, $H_0^{(1)}(\cdot)$ and $H_1^{(1)}(\cdot)$ are Hankel functions of the first kind,

$$k_L = k_T\sqrt{\mu/(\lambda+2\mu)}, \quad \alpha_n = n\pi/h,$$

and

$$\zeta_{nT,L} = \begin{bmatrix} \sqrt{\alpha_n^2-k_{T,L}^2}, & \alpha_n > k_{T,L} \\ -i\sqrt{k_{T,L}^2-\alpha_n^2}, & \alpha_n < k_{T,L}. \end{bmatrix} \tag{8}$$

Following the approach adopted by Meade and Keer, the interaction problem is decomposed into diffraction and radiation problems. This approach was introduced, apparently, by Thau (1967). As indicated by Thau, the diffraction problem pertains to interaction of the incident wave with immobilized strips, whereas the radiation problem pertains to wave generation in the surrounding elastic medium due to rigid body motion of the strips. It is clear that both the diffraction and radiation problems are defined by eqn (4). Now if the displacement components of the diffracted and radiated fields are denoted as (u_1, v_1) and (u_2, v_2), respectively, then the boundary conditions on $y = 0$, $|x| < 1$ are

$$u_1 = 0, \quad v_1 = -v_i = -e^{-ik_Tx} \tag{9}$$

$$u_2 = 0, \quad v_2 = \Delta + x\theta, \tag{10}$$

where Δ and θ are the (unknown) rigid body translation and rotation, respectively, of the basic strip. The first of conditions (9) and (10), it is noted, has been satisfied already by condition (3). The second of conditions (9) and (10) is handled most simply by a decomposition into symmetric and antisymmetric components.

SYMMETRIC CASE

For the symmetric case $v_j(-s) = v_j(s)$ and $[p_j(-s)] = [p_j(s)]$, $j = 1,2$. Putting $[p_j(s)] = 1/2\, \partial f_j/\partial s$ and

$$f_j(s) = sf_j(1) + \sqrt{1-s^2}\, g_j(s) \tag{11}$$

into (4), and integrating by parts, leads to

$$-4v_j(x) = 1/\pi \int_{-1}^{1} \sqrt{1-s^2}\, g_j(s)[\gamma/w + L(s,x)]\, ds$$
$$+ f_j(1)\left[i/2 \int_{-1}^{1} P(|w|)\, ds - Q(x)\right] \quad |x| < 1, \tag{12}$$

where $\gamma = (1+\kappa)/2\kappa$, $\kappa = (\lambda+2\mu)/\mu$, and

$$L(s,x) = i\pi/2k_T \operatorname{sgn} wN(|w|) + \pi \operatorname{sgn} wM(|w|), \tag{13}$$

$$N(|w|) = H_1^{(1)}(k_T|w|) - (k_T|w|)^{-1}H_2^{(1)}(k_T|w|) + (\kappa k_T|w|)^{-1}H_2(k_L|w|) + 2i\gamma(\pi k_T|w|)^{-1}, \tag{14}$$

$$P(|w|) = H_0^{(1)}(k_T|w|) - (k_T|w|)^{-1}H_1^{(1)}(k_T|w|) + (\sqrt{\kappa}k_T|w|)^{-1}H_1^{(1)}(k_L|w|), \tag{15}$$

$$Q(x) = (hk_T^2)^{-1}\sum_1^{\infty}[(\alpha_n/\zeta_{nL})^2(e^{-\zeta_{nL}(1-x)} + e^{-\zeta_{nL}(1+x)}) - (e^{-\zeta_{nT}(1-x)} + e^{-\zeta_{nT}(1+x)})], \tag{16}$$

$$M(|w|) = (hk_T^2)^{-1} \sum_1^L [\alpha_n^2 e^{-\zeta_{nL}|w|} - \zeta_{nT}^2 e^{-\zeta_{nT}|w|}], \tag{17}$$

and sgn $w = +1(-1)$ if $w > 0(<0)$. The quantities $f_j(1)$, $j = 1, 2$ are proportional to the resultant (vertical) forces F_j acting on the basic strip, i.e.

$$F_j = \mu \int_{-1}^{1} [p_j(s)] \, ds = \mu f_j(1). \tag{18}$$

In addition, the left hand side of (12) is

$$v_j(x) = \begin{bmatrix} -\cos k_T x, & j = 1 \\ \Delta, & j = 2 \end{bmatrix} \quad |x| < 1. \tag{19}$$

The unknown Δ is determined by letting $v_2'(x) = v_2(x)/\Delta$ and solving (12) with $v_2'(x) = 1$. This gives $g_2'(s) = g_2(s)/\Delta$, $f_2'(1) = f_2(1)/\Delta$, and $F_2' = F_2/\Delta$. Then, using the dynamic (force) equilibrium condition, gives

$$\Delta = -F_1/(F_2' + \mu m k_T^2), \tag{20}$$

where $m = 2\rho^*/\rho$.

The solution of (12) may be obtained by the Gauss Chebyshev numerical technique (see e.g. Erdogan and Gupta, 1972). According to this technique equation (12) is replaced by the discrete system

$$-4v_{jp} = 1/(N+1) \sum_1^N (1 - s_q^2) g_{jq} (\gamma/w_{qp} + L_{qp})$$
$$+ f_j(1) \left[i/2 \int_{-1}^{1} P(|w_p|) \, ds - Q_p \right], \quad p = 1, 2, \ldots, N+1, \tag{21}$$

where $v_{jp} = v_j(x_p)$, $w_{qp} = s_q - x_p$, $w_p = s - x_p$, etc., $x_p = \cos[\pi(2p-1)/2(N+1)]$, and $s_q = \cos \pi q/(N+1)$.

ANTISYMMETRIC CASE

For the antisymmetric case $v_j(-s) = -v_j(s)$ and $[p_j(-s)] = -[p_j(s)]$, $j = 1, 2$. Using the same notation as in the symmetric case, (11) is replaced by

$$f_j(s) = |s| f_j(1) + \sqrt{1 - s^2} g_j(s), \tag{22}$$

so that (12) becomes

$$-4v_j(x) = 1/\pi \int_{-1}^{1} \sqrt{1 - s^2} g_j(s) [\gamma/w + L(s, x)] \, ds$$
$$+ f_j(1) \left[i/2 \int_{-1}^{1} P(|w|) \operatorname{sgn} s \, ds - R(x) \right] \quad |x| < 1, \tag{23}$$

where

$$R(x) = (hk_T^2)^{-1} \sum_1^T [(\alpha_n/\zeta_{nL})^2 (e^{-\zeta_{nL}(1-x)} - e^{-\zeta_{nL}(1+x)} + 2e^{-\zeta_{nL}|x|} \operatorname{sgn} x)$$
$$- (e^{-\zeta_{nT}(1-x)} - e^{-\zeta_{nT}(1+x)} + 2 e^{-\zeta_{nT}|x|} \operatorname{sgn} x)]. \tag{24}$$

Here the left hand side is

$$v_j(x) = \begin{bmatrix} i \sin k_T x, & j = 1 \\ x\theta, & j = 2 \end{bmatrix}, \quad |x| < 1. \tag{25}$$

As in the symmetric case the rotation θ is obtained by solving equation (23) with $\theta = 1$ and using the dynamic (moment) equilibrium equation. Thus, with the notation $g_2(s) = \theta g'_2(s)$, $f_2(1) = \theta f'_2(1)$, and $T_2 = \theta T'_2$, there follows

$$\theta = -T_1/(T'_2 + \mu p k_T^2), \tag{26}$$

where $p = I/\rho$ and I is the (mass) moment of inertia of the strip. (Here $I = \frac{1}{3}\rho^*$.) Combining the basic definition of torque

$$T_j = \mu \int_{-1}^{1} [p_j(s)] s \, ds, \tag{27}$$

with equation (22) yields

$$T_j = \mu/2 \left[f_j(1) - \int_{-1}^{1} \sqrt{1-s^2} g_j(s) \, ds \right]. \tag{28}$$

The second term is evaluated by the quadrature formula

$$\int_{-1}^{1} \sqrt{1-s^2} g_j(s) \, ds \approx \frac{\pi}{N+1} \sum_{1}^{N} (1-s_q^2) g_{jq}, \tag{29}$$

where g_{jq} is the solution of the discretized form of equation (23). For the antisymmetric case the discretized equations are:

$$-4v_{jp} = 1/(N+1) \sum_{1}^{N} (1-s_q^2) g_{jq} [\gamma/w_{qp} + L_{qp}]$$
$$+ f_j(1) \left[i/2 \int_{-1}^{1} P(|w_p|) \operatorname{sgn} s \, ds - R_p \right], \quad p = 1, 2, \ldots, N+1 \tag{30}$$

where $R_p = R(x_p)$ is given by (24), and all other terms are as previously defined.

RESULTS AND DISCUSSION

Equations (21) and (30) were solved for the case $\kappa = 3$ (Poisson's ratio $\nu = 1/4$). A value of $N = 20$ was found to give stable results over the entire frequency range considered. Figures 2–6 depict the displacement $|\Delta|$ as a function of wavenumber k_T for various spacings h and mass ratios m. (Note that since the unit of length is the strip half-width, k_T is really a normalized wavenumber.) As seen, an increase in mass ratio causes a decrease in strip displacement. An increase in strip spacing, however, tends to magnify the displacement at certain ("resonant") wavenumbers. Note that the locations of the "resonant" peaks are practically independent of m, whereas the magnitudes of the "resonant" peaks are not. In addition to the results shown, some computations were made with $h = 10$. These latter results differed only slightly from those with $h = 5$. It is expected, therefore, that the case $h = 5$ represents fairly closely the case of a single strip ($h = \infty$).

It is of interest to compare the "resonant" wavenumbers with those corresponding to a column of identical material 2 units in width by $2h$ units in height, and with a strip mass at its midpoint. The natural wavenumbers for longitudinal vibrations of the column are

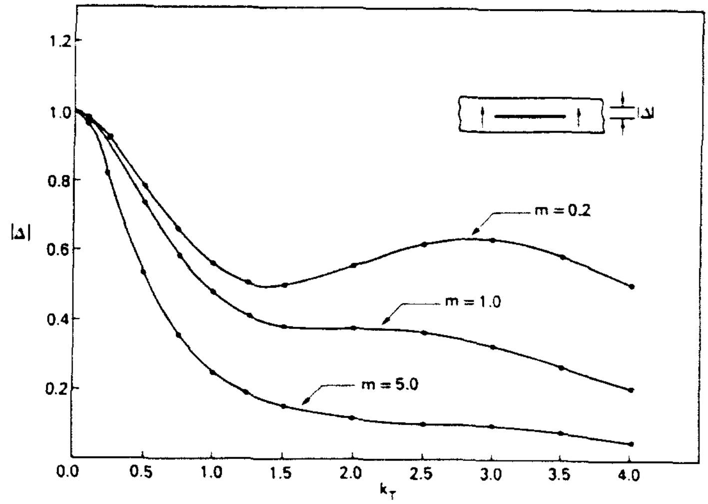

Fig. 2. Displacement vs normalized wavenumber: $h = 0.5$.

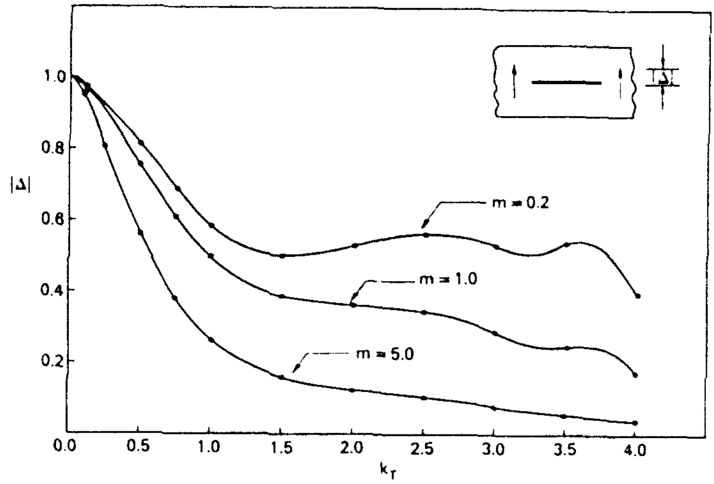

Fig. 3. Displacement vs normalized wavenumber: $h = 1.0$.

given by

$$\tan k_L h = -mk_L, \tag{31}$$

where, as previously defined, $k_L = k_T/\sqrt{\kappa}$ and $m = 2\rho^*/\rho$. With $h = 5$, $\kappa = 3$, and $m = 0.2$, the first three roots of (31) are found to agree to within 2% on the average with those of Fig. 4. The agreement is not as good, however, at the higher mass ratios; differences of about 10% and 20%, respectively, are obtained for $m = 1.0$ and $m = 5.0$. Apparently, a less localized interaction occurs as the mass ratio increases. We note that the results depicted in Figs 2 and 3 are similar to those depicted in Figs 5 and 6 of Meade and Keer. The "resonant" effect, however, does not occur in the case considered by Meade and Keer.

The rotational response of the strips is shown in Figs 7 and 8. In these figures the parameter p is proportional to the mass ratio m, i.e. $p = I/\rho = 2m/3$. It is seen that the effect of spacing on the rotational response is appreciable only when the strips are relatively dense. It is of interest to compare the effects of the spacing and mass parameters on the rotational and translational responses, especially the "resonant" responses. From Figs 2–6 it is seen that a "resonant" displacement occurs only when the spacing is large, irrespective of the strip mass. By contrast, Figs 7 and 8 reveal that a "resonant" rotation

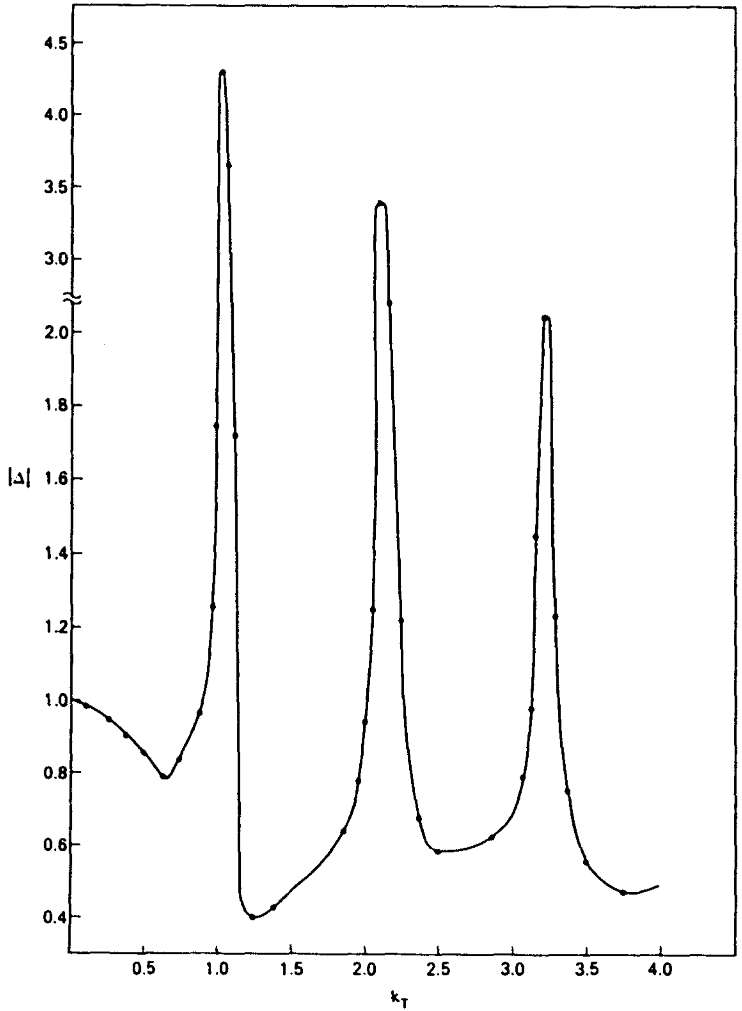

Fig. 4. Displacement vs normalized wavenumber: $h = 5.0$, $m = 0.2$.

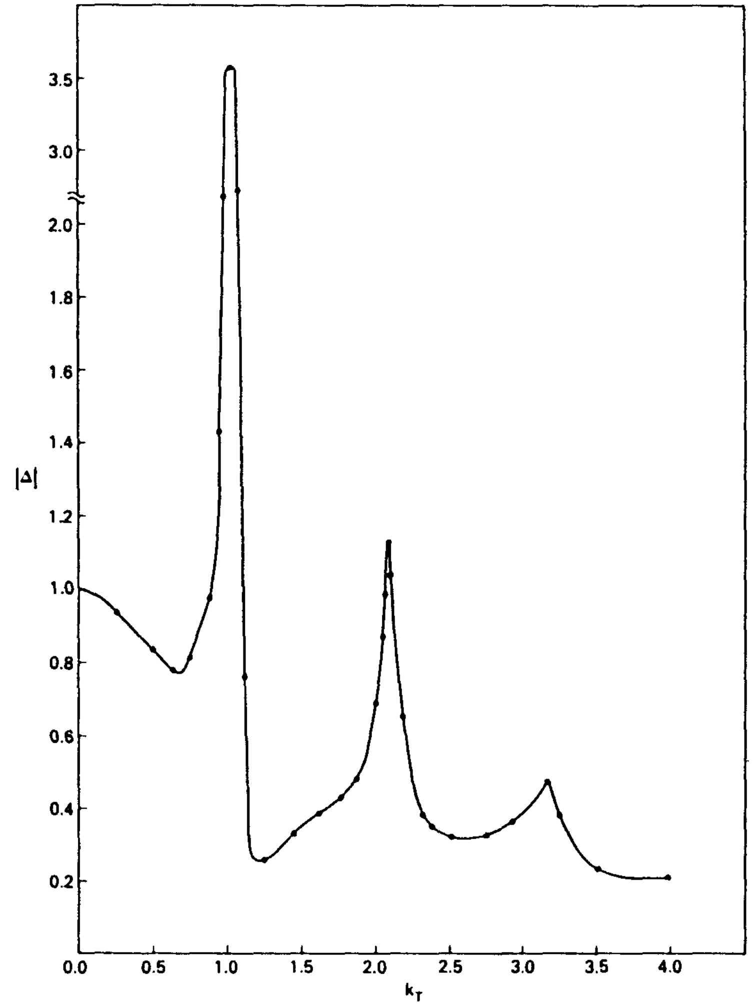

Fig. 5. Displacement vs normalized wavenumber: $h = 5.0$, $m = 1.0$.

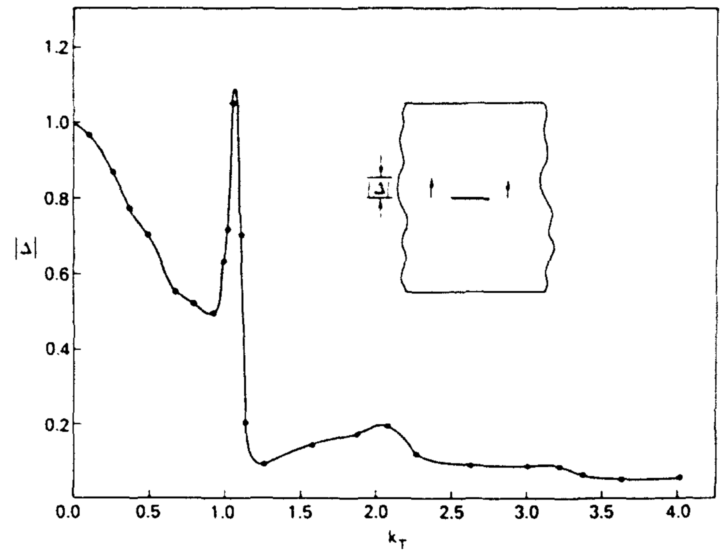

Fig. 6. Displacement vs normalized wavenumber: $h = 5.0$, $m = 5.0$.

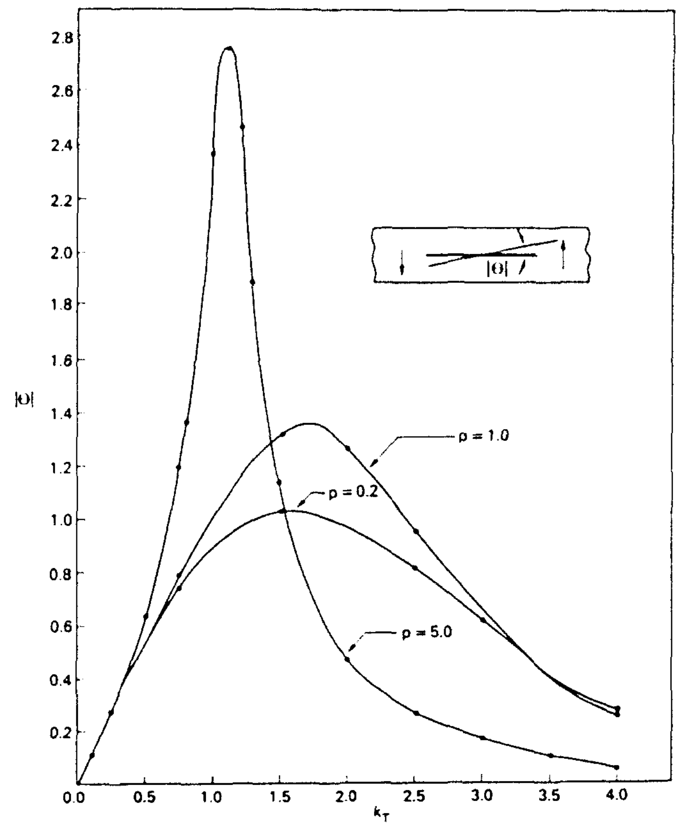

Fig. 7. Rotation vs normalized wavenumber: $h = 0.5$.

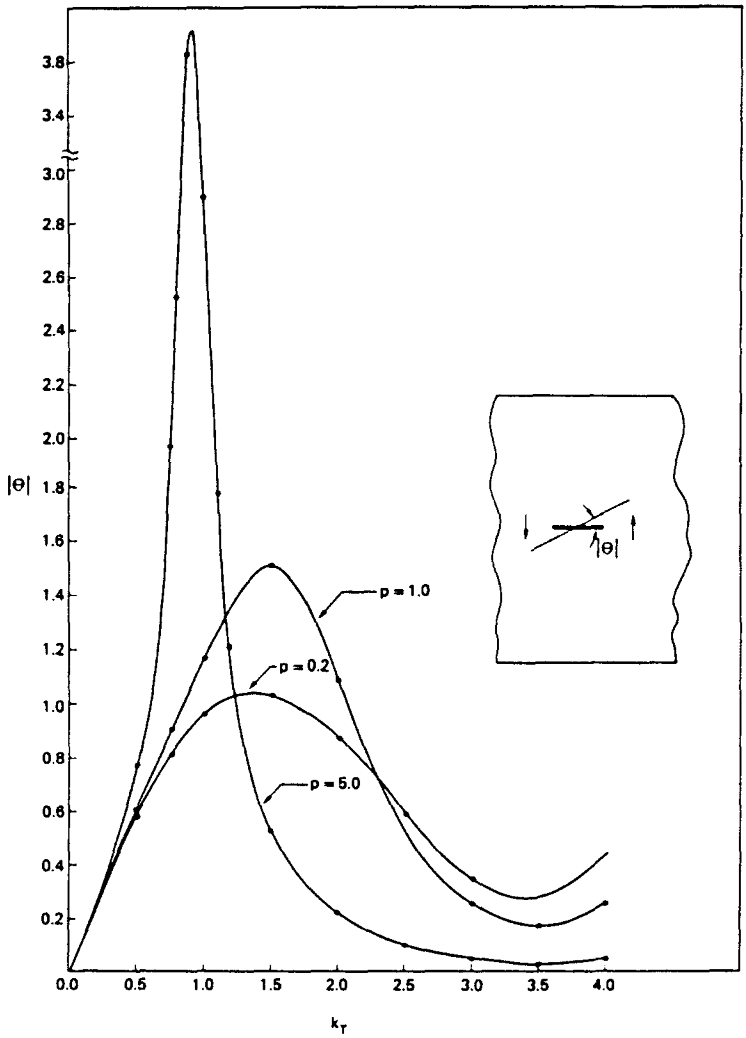

Fig. 8. Rotation vs normalized wavenumber: $h = 5.0$.

occurs only when the mass ratio is large, irrespective of the spacing. That is, the effects of the mass and spacing parameters on the "resonant" rotation and "resonant" displacement are reversed.

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